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Regularity and Explicit Representation of $(0, 1, ..., m-2, m)$-Interpolation on the Zeros of the Jacobi Polynomials*

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A necessary and sufficient condition of regularity of $(0, 1, ..., m-2, m)$-interpolation on the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geqslant -1)$ in a manageable form is established. Meanwhile, the explicit representation of the fundamental polynomials, when they exist, is given. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let us consider a system A of nodes

$$1 \geqslant x_1 > x_2 > \cdots > x_n \geqslant -1, \qquad n \geqslant 2. \tag{1.1}$$

Let $\mathbf{P}_n$ be the set of polynomials of degree at most n and let $m \geqslant 2$ be a fixed integer. The problem of $(0, 1, ..., m-2, m)$-interpolation is, given a set of numbers,

$$y_{kj}, \qquad k \in N := \{1, 2, ..., n\}, \quad j \in M := \{0, 1, ..., m-2, m\} \tag{1.2}$$

to determine a polynomial $R_{mn-1}(x; A) \in \mathbf{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k; A) = y_{kj}, \qquad \forall k \in N, \forall j \in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1}(x; A) \in \mathbf{P}_{mn-1}$ satisfying (1.3) then we say that the problem of $(0, 1, ..., m-2, m)$-interpolation on A is regular (otherwise, is singular) and $R_{mn-1}(x; A)$ can be uniquely written as [2]

$$R_{mn-1}(x; A) = \sum_{\substack{k \in N \\ j \in M}} y_{kj} r_{kj}(x; A), \tag{1.4}$$

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where $r_{kj} \in \mathbf{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \qquad k, \nu \in N, \qquad j, \mu \in M \tag{1.5}$$

and are called the fundamental polynomials.

In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \qquad k = 1, 2, ..., n. \tag{1.6}$$

On the problem of (0, 2)-interpolation Turán in [3] raises an open problem as follows.

Problem 29. Find all Jacobi matrices $P(\alpha, \beta)$, $\alpha \neq \beta$, for which the (0, 2)-interpolation problem does have a unique solution.

By a Jacobi matrix $P(\alpha, \beta)$, Turán means the triangular matrix whose nth row consists of the zeros of the nth Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geqslant -1)$.

Recently, Chak, Sharma, and Szabados [1] have given a necessary and sufficient condition of regularity of (0, 2)-interpolation in a manageable form on all Jacobi matrices $P(\alpha, \beta)$.

THEOREM A. *The problem of* (0, 2)*-interpolation on the zeros of* $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$ *is uniquely solvable if and only if*

$$D_n^*(\alpha, \beta) \neq 0, \tag{1.7}$$

where

$$D_n^*(\alpha, \beta) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\binom{(1/2)(\alpha+1)}{k}\binom{(1/2)(\beta+1)}{n-k}}{\binom{n}{k}}. \tag{1.8}$$

When $\alpha = -1$, $\beta > -1$ *the problem is always uniquely solvable.*

Meanwhile in their nice paper they also give the explicit representation of the fundamental polynomials when they exist, except for the case when $\alpha, \beta > -1$ and $\alpha + \beta = \text{even}$ (see [1, Theorem 2]).

Following the main idea of Chak, Sharma, and Szabados in [1] in this paper we attempt to give a necessary and sufficient condition of regularity of $(0, 1, ..., m-2, m)$-interpolation for all Jacobi matrices $P(\alpha, \beta)$ $(\alpha, \beta \geqslant -1)$. Meanwhile, we will also give the explicit representation of the fundamental polynomials when they exist without exception. Thus our results improve and extend the ones of [1].

In Section 2 we state some needed properties for the Jacobi polynomials. In Section 3, a useful lemma of regularity for general $(0, 1, ..., m-2, m)$-interpolation is given. Section 4 deals with the regularity for $P(\alpha, \beta)$ and the explicit representation for the fundamental polynomials.

2. Preliminaries

The following results are taken from [1, (2.1)–(2.4)].

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, α, $\beta \geqslant -1$, satisfies the differential equation

$$(1-x^2)\,y'' + [(\beta-\alpha)-(\beta+\alpha+2)x]\,y' + n(n+\alpha+\beta+1)\,y = 0 \tag{2.1}$$

and the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{2.2}$$

We have

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k} (x-1)^k\,(x+1)^{n-k}, \tag{2.3}$$

$$P_n^{(-1,\beta)}(x) = \frac{n+\beta}{2n}\,(x-1)\,P_{n-1}^{(1,\beta)}(x). \tag{2.4}$$

Now using (2.3) we can get

$$P_n^{(-1,-1)}(x) = \frac{1}{2n}\,(x^2-1)\,P'_{n-1}(x). \tag{2.5}$$

3. An Auxiliary Lemma

We first prove a lemma which is of independent interest. To this end we introduce the fundamental polynomials of $(0, 1, ..., m-1)$-interpolation. Let A_{kj}, $B_k \in \mathbf{P}_{mn-1}$ be defined by

$$A_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \qquad k, \nu = 1, 2, ..., n, \qquad j, \mu = 0, 1, ..., m-1 \tag{3.1}$$

and

$$B_k(x) := A_{k,m-1}(x) = \frac{1}{m!}\,(x-x_k)^{m-1}\,l_k^m(x), \qquad k = 1, 2, ..., n, \tag{3.2}$$

where

$$l_k(x) := \frac{\omega_n(x)}{(x-x_k)\,\omega_n'(x_k)}, \qquad \omega_n(x) = c(x-x_1)(x-x_2)\cdots(x-x_n)\ (c \neq 0). \tag{3.3}$$

Then we have

LEMMA. *If there is one index i, $1 \leqslant i \leqslant n$, such that $\rho_i(x) \in \mathbf{P}_{mn-1}$ with the properties* (1.5) *exists uniquely, then the problem of* $(0, 1, ..., m-2, m)$-*interpolation is regular and*

$$r_{kj}(x) = A_{kj}(x) - \sum_{\nu=1}^{n} A_{kj}^{(m)}(x_\nu)\,\rho_\nu(x), \qquad k = 1, 2, ..., n, \qquad j = 0, 1, ..., m-2. \tag{3.4}$$

Proof. Since the problem of $(0, 1, ..., m-1)$-interpolation is always regular and ρ_i has a unique solution, ρ_i can be uniquely written as

$$\rho_i(x) = \sum_{\nu=1}^{n} \sum_{\mu=0}^{m-1} \rho_i^{(\mu)}(x_\nu)\,A_{\nu\mu}(x) = \sum_{\nu=1}^{n} \rho_i^{(m-1)}(x_\nu)\,B_\nu(x). \tag{3.5}$$

This shows that the system of equations

$$\sum_{\nu=1}^{n} \rho_i^{(m-1)}(x_\nu)\,B_\nu^{(m)}(x_k) = \delta_{ik}, \qquad k = 1, 2, ..., n \tag{3.6}$$

has a unique solution, which is equivalent to the nonsingularity of the coefficient matrix

$$\mathbf{B}_n := [B_\nu^{(m)}(x_k)]_{\nu,k=1}^{n}. \tag{3.7}$$

Thus for an arbitrary set of numbers y_{kj} if we put

$$R_{mn-1}(x) = \sum_{\nu=1}^{n} \sum_{\mu=0}^{m-1} a_{\nu\mu} A_{\nu\mu}(x)$$

then the system of equations (1.3) becomes

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m-1} a_{\nu\mu} A_{\nu\mu}^{(j)}(x_k) = y_{kj}, \qquad k \in N, \qquad j \in M.$$

Hence we obtain

$$a_{kj} = y_{kj}, \qquad k = 1, 2, ..., n, \qquad j = 0, 1, ..., m-2$$

and then the system of equations

$$\sum_{\nu=1}^{n} a_{\nu,m-1} B_{\nu}^{(m)}(x_k) = y_{km} - \sum_{\nu=1}^{n} \sum_{\mu=0}^{m-2} y_{\nu\mu} A_{\nu\mu}^{(m)}(x_k), \qquad k = 1, 2, ..., n$$

must have a unique solution, because the matrix $\mathbf{B}_n$ is nonsingular. This means that the problem is regular.

Finally it is easy to see that (3.4) is true when r_{kj} satisfies (1.5).

This completes the proof.

Remark. Since the explicit representation for the A_{kj}'s is well known, by (3.4) it is sufficient to find the one for the ρ_k's.

4. Main Results

In what follows let n be fixed and (1.1) the zeros of $P_n^{(\alpha,\beta)}(x)$. Write

$$\gamma := \tfrac{1}{2}(m-1)(\alpha+1), \qquad \delta := \tfrac{1}{2}(m-1)(\beta+1), \tag{4.1}$$

$$\gamma_k := 2^{-n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}, \qquad k = 0, 1, ..., n. \tag{4.2}$$

The main result in this paper is the following

THEOREM. *The problem of* $(0, 1, ..., m-2, m)$ *interpolation on the zeros of* $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta \geqslant -1)$ *is regular if and only if*

$$D_n(\alpha, \beta) \neq 0, \tag{4.3}$$

where

$$D_n(\alpha,\beta) = \begin{cases} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k \binom{\gamma}{k}\binom{\delta}{n-k}\gamma_k}{\binom{n}{k}}, & \alpha, \beta > -1 \\ (m+1)\binom{\delta}{n} - (m-1)\binom{n+\beta+\delta}{n}, & \alpha = -1, \beta > -1 \\ (-1)^n D_n(-1, \alpha), & \alpha > -1, \beta = -1 \\ 1 + (-1)^n, & \alpha = \beta = -1. \end{cases} \tag{4.4}$$

In particular, when $\alpha = -1$, $\beta > -1$ *or* $\alpha > -1$, $\beta = -1$ *the problem is always regular; when* $\alpha = \beta = -1$ *the problem is regular for even* n *and singular for odd* n.

If the problem is regular, then for each i, $1 \leqslant i \leqslant n$, the fundamental polynomial $\rho_i(x) := \rho_i(x; \alpha, \beta)$ is given by

$$\rho_i(x; \alpha, \beta) = \begin{cases} (-1)^m \rho_{n+1-i}(-x; \beta, \alpha), & \alpha > -1, \beta = -1 \\ [P_n^{(\alpha,\beta)}(x)]^{m-1} q_i(x), & \text{otherwise}, \end{cases} \tag{4.5}$$

in which $q_i \in \mathbf{P}_{n-1}$ is of the form

$$q_i(x) = (1-x)^\gamma (1+x)^\delta \times \left\{ d_i + \int_a^x [Q_i(t) - c_i P_n^{(\alpha,\beta)}(t)](1-t)^{-\gamma-1} (1+t)^{-\delta-1} \, dt \right\} \tag{4.6}$$

with certain constants d_i and c_i, where

$$a = \begin{cases} 0, & \alpha, \beta > -1 \\ 1, & \alpha = -1, \end{cases} \tag{4.7}$$

$$Q_i(x) = \frac{(1-x_i^2)\, l_i(x)}{m!\, [P_n'^{(\alpha,\beta)}(x_i)]^{m-1}}. \tag{4.8}$$

Proof. For simplicity we write

$$\omega_n(x) = P_n^{(\alpha,\beta)}(x). \tag{4.9}$$

By (3.5) and (3.2) we may set

$$\rho_i(x) = \omega_n^{m-1}(x)\, q_i(x), \tag{4.10}$$

where $q_i \in \mathbf{P}_{n-1}$. Then the requirement (1.5) yields

$$[\omega_n^{m-1}(x)\, q_i(x)]^{(m)}_{x=x_k} = \delta_{ik}, \qquad k = 1, 2, ..., n. \tag{4.11}$$

It is easy to see that

$$[\omega_n^{m-1}(x)]^{(m)}_{x=x_k} = \tfrac{1}{2}(m-1)\, m!\, \omega_n'(x_k)^{m-2}\, \omega_n''(x_k)$$

and

$$[\omega_n^{m-1}(x)]^{(m-1)}_{x=x_k} = (m-1)!\, \omega_n'(x_k)^{m-1}.$$

Then (4.11) becomes

$$\frac{1}{2}(m-1)\, \omega_n''(x_k)\, q_i(x_k) + \omega_n'(x_k)\, q_i'(x_k) = \frac{\delta_{ik}}{m!\, \omega_n'(x_k)^{m-2}}, \qquad k = 1, 2, ..., n. \tag{4.12}$$

It follows from (2.1) that

$$(1-x_k^2)\,\omega_n''(x_k)=[(\alpha+1)(1+x_k)-(\beta+1)(1-x_k)]\,\omega_n'(x_k),$$
$$k=1,2,...,n. \tag{4.13}$$

This, coupled with (4.12), gives

$$(1-x_k^2)\,q_i'(x_k)+[\gamma(1+x_k)-\delta(1-x_k)]\,q_i(x_k)$$
$$=\frac{(1-x_k^2)\,\delta_{ik}}{m!\,\omega_n'(x_k)^{m-1}},\qquad k=1,2,...,n. \tag{4.14}$$

Denote by **D** the differential operator

$$\mathbf{D}y:=(1-x^2)\,y'+[\gamma(1+x)-\delta(1-x)]\,y. \tag{4.15}$$

Then (4.14) implies

$$\mathbf{D}q_i(x)=Q_i(x)-c_i\omega_n(x), \tag{4.16}$$

where c_i is a constant to be determined and $Q_i(x)$ is given by (4.8). Solving this differential equation we get (4.6) with a constant d_i to be determined.

Now let us determine c_i and d_i. To this end put

$$q_i(x)=\sum_{k=0}^{n-1}\alpha_k(x-1)^k\,(x+1)^{n-1-k}. \tag{4.17}$$

Meanwhile we write

$$Q_i(x)=\sum_{k=0}^{n}\beta_k(x-1)^k\,(x+1)^{n-k}. \tag{4.18}$$

We distinguish four cases.

Case I ($\alpha,\beta>-1$). Using (4.17), (4.18), and (2.3), and comparing the coefficients of $(x-1)^k\,(x+1)^{n-k}$ on both sides in (4.16) we obtain the system of equations

$$\begin{cases}(\delta-n+k)\,\alpha_{k-1}+(\gamma-k)\alpha_k+\gamma_kc_i=\beta_k & (k=0,1,...,n)\\ \alpha_{-1}=\alpha_n=0.\end{cases} \tag{4.19}$$

Expanding the coefficient determinant of this system in terms of the elements of the last column we get (4.4) except for a nonzero factor. We know that this system has a unique solution if and only if (4.3) is true. By the Lemma this is equivalent to the regularity of $(0,1,...,m-2,m)$-interpolation.

Solving (4.19) by Cramer's rule we get c_i.

If $\gamma \neq$ an integer or $k < \gamma$ then by (4.19)

$$\alpha_k = \frac{1}{\gamma - k} \{(n - \delta - k)\, \alpha_{k-1} + \beta_k - \gamma_k c_i\}$$

and hence by induction we get the formula of α_k. Similarly if $\delta \neq$ an integer or $k \geqslant n - \delta$ then it follows from (4.19) that

$$\alpha_k = \frac{1}{\delta - n + k + 1} \{(k + 1 - \gamma)\, \alpha_{k+1} + \beta_{k+1} - \gamma_{k+1} c_i\}$$

and hence by induction we also get the formula of α_k. Then we can determine d_i, since in this case by (4.6) and (4.17) one has

$$d_i = q_i(0) = \sum_{k=0}^{n-1} (-1)^k \alpha_k.$$

We point out that if (4.3) is true then α_k may always be determined. In fact, if both γ and δ are integers, and if for some k, $1 \leqslant k \leqslant n$, the inequalities $k \geqslant \gamma$ and $n - k > \delta$ hold, then $n > \gamma + \delta$. Thus for each j, $0 \leqslant j \leqslant n$, either $j > \gamma$ or $n - j > \delta$ holds and hence $\binom{\gamma}{j}\binom{\delta}{n-j} = 0$, which implies $D_n(\alpha, \beta) = 0$, a contradiction.

Case II ($\alpha = -1$, $\beta > -1$). In this case $x_1 = 1$ and $\gamma = \gamma_0 = \beta_0 = 0$. Then the equation with $k = 0$ in (4.19) becomes an identity. But by (2.4) we have

$$P_n'^{(-1,\beta)}(1) = \frac{n+\beta}{2n} P_{n-1}^{(1,\beta)}(1) = \frac{1}{2}(n+\beta) \tag{4.20}$$

and

$$P_n''^{(-1,\beta)}(1) = \frac{n+\beta}{n} P_{n-1}'^{(1,\beta)}(1). \tag{4.21}$$

Thus by (4.12) one has

$$\frac{(m-1)}{n} P_{n-1}'^{(1,\beta)}(1)\, q_i(1) + q_i'(1) = \frac{2^{m-1}\delta_{i1}}{m!\,(n+\beta)^{m-1}}. \tag{4.22}$$

On the other hand, by (2.1), $P_{n-1}^{(1,\beta)}(x)$ satisfies the equation

$$(1 - x^2)\, P_{n-1}''^{(1,\beta)}(x) + [(\beta - 1) - (\beta + 3)x]\, P_{n-1}'^{(1,\beta)}(x) + (n-1)(n+\beta+1)\, P_{n-1}^{(1,\beta)}(x) = 0 \tag{4.23}$$

and hence by (2.2)

$$P_{n-1}'^{(1,\beta)}(1) = \tfrac{1}{4}(n-1)(n+\beta+1)\, P_{n-1}^{(1,\beta)}(1) = \tfrac{1}{4} n(n-1)(n+\beta+1). \tag{4.24}$$

This, together with (4.22), gives

$$4q_i'(1)+(m-1)(n-1)(n+\beta+1)\,q_i(1)=\frac{2^{m+1}\delta_{i1}}{m!\,(n+\beta)^{m-1}}. \tag{4.25}$$

Meanwhile, by means of (4.17) we obtain

$$q_i'(1)=2^{n-2}[(n-1)\,\alpha_0+\alpha_1] \tag{4.26}$$

and

$$q_i(1)=2^{n-1}\alpha_0. \tag{4.27}$$

Therefore (4.25) becomes

$$\alpha_1+\frac{1}{2}(n-1)[(m-1)n+2\delta+2]\,\alpha_0=\frac{2^{m+1-n}\delta_{i1}}{m!\,(n+\beta)^{m-1}}. \tag{4.28}$$

Adding this equation to the equation with $k=1$ in (4.19) we get

$$\frac{1}{2}(m-1)(n+\beta)\,n\alpha_0+\gamma_1 c_i=\beta_1+\frac{2^{m+1-n}\delta_{i1}}{m!\,(n+\beta)^{m-1}}. \tag{4.29}$$

At last we obtain the system of equations for this case:

$$\begin{cases}\frac{1}{2}(m-1)(n+\beta)\,n\alpha_0+\gamma_1 c_i=\beta_1+\frac{2^{m+1-n}\delta_{i1}}{m!\,(n+\beta)^{m-1}}\\ (\delta-n+k)\,\alpha_{k-1}-k\alpha_k+\gamma_k c_i=\beta_k \qquad (k=1,\ldots,n)\\ \alpha_n=0.\end{cases} \tag{4.30}$$

Expanding the coefficient determinant of this system in terms of the elements of the last column we get

$$\begin{aligned}A_n(-1,\beta)&=\binom{\delta}{n}n!\,\gamma_1+\frac{1}{2}(m-1)(n+\beta)n\\ &\quad\times\sum_{k=1}^{n}(-1)^k\gamma_k(-1)^{k-1}(k-1)!\binom{\delta}{n-k}(n-k)!\\ &=2^{-n-1}n!\,(n+\beta)\left\{2\binom{\delta}{n}-(m-1)\sum_{k=1}^{n}\binom{n+\beta}{k}\binom{\delta}{n-k}\right\}\\ &=2^{-n-1}n!\,(n+\beta)\left\{(m+1)\binom{\delta}{n}-(m-1)\sum_{k=0}^{n}\binom{n+\beta}{k}\binom{\delta}{n-k}\right\}\\ &=2^{-n-1}n!\,(n+\beta)\left\{(m+1)\binom{\delta}{n}-(m-1)\binom{n+\beta+\delta}{n}\right\}\\ &=2^{-n-1}n!\,(n+\beta)\,D_n(-1,\beta),\end{aligned}$$

here we use an identity [1, p. 446]

$$\sum_{k=0}^{n} \binom{n+\beta}{k}\binom{\delta}{n-k} = \binom{n+\beta+\delta}{n}.$$

Since

$$(m-1)\binom{n+\beta+\delta}{n} = \frac{(m^2-1)(\beta+1)}{2n!} \prod_{k=1}^{n-1} \left[\frac{1}{2}(m+1)(\beta+1)+k\right]$$

and

$$(m+1)\binom{\delta}{n} = \frac{(m^2-1)(\beta+1)}{2n!} \prod_{k=1}^{n-1} \left[\frac{1}{2}(m-1)(\beta+1)-k\right],$$

it follows from

$$\tfrac{1}{2}(m+1)(\beta+1)+k > \tfrac{1}{2}(m-1)(\beta+1)+k > |\tfrac{1}{2}(m-1)(\beta+1)-k|$$

that $D_n(-1, \beta) \neq 0$.

Now solving (4.30) we can determine c_i.

Clearly, in this case by (4.6) and (4.27) we have

$$d_i = 2^{-\delta} q_i(1) = 2^{n-1-\delta}\alpha_0.$$

This, coupled with the first equation in (4.30), gives d_i.

Case III ($\alpha > -1, \beta = -1$). First we note that

$$P_n^{(\beta,\alpha)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x). \tag{4.31}$$

Using (4.31) and the above arguments we obtain the formulas

$$\begin{aligned}
P_n'^{(\alpha,-1)}(-1) &= \tfrac{1}{2}(-1)^{n-1}(n+\alpha),\\
P_n''^{(\alpha,-1)}(-1) &= \tfrac{1}{4}(-1)^n (n-1)(n+\alpha)(n+\alpha+1),\\
q_i(-1) &= (-2)^{n-1}\alpha_{n-1},\\
q_i'(-1) &= (-2)^{n-2}[(n-1)\alpha_{n-1}+\alpha_{n-2}].
\end{aligned}$$

Then it follows from (4.12) and the equation with $k=n-1$ in (4.19) that

$$\frac{1}{2}(m-1)(n+\alpha)\, n\alpha_{n-1} + \gamma_{n-1}c_i = \beta_{n-1} + \frac{(-1)^{m(n-1)+1}\, 2^{m+1-n}\delta_{in}}{m!\,(n+\alpha)^{m-1}}. \tag{4.32}$$

At last we obtain the system of equations for this case:

$$\begin{cases} (k-n)\,\alpha_{k-1}+(\gamma-k)\,\alpha_k+\gamma_k c_i=\beta_k \qquad (k=0, 1, ..., n-1) \\ \dfrac{1}{2}(m-1)(n+\alpha)\,n\alpha_{n-1}+\gamma_{n-1}c_i=\beta_{n-1}+\dfrac{(-1)^{m(n-1)+1}\,2^{m+1-n}\delta_{in}}{m!\,(n+\alpha)^{m-1}} \\ \alpha_{-1}=0. \end{cases}$$

By calculation we get

$$D_n(\alpha, -1)=(-1)^n\,D_n(-1, \alpha).$$

By (4.31) and by the definition of the ρ_i's it is easy to check the first formula in (4.5).

Case IV ($\alpha=\beta=-1$). In this case (4.29) and (4.32) become

$$\frac{1}{2}(m-1)(n-1)\,n\alpha_0+\gamma_1 c_i=\beta_1+\frac{2^{m+1-n}\delta_{i1}}{m!\,(n-1)^{m-1}} \tag{4.33}$$

and

$$\frac{1}{2}(m-1)(n-1)\,n\alpha_{n-1}+\gamma_{n-1}c_i=\beta_{n-1}+\frac{(-1)^{m(n-1)+1}\,2^{m+1-n}\delta_{in}}{m!\,(n-1)^{m-1}}. \tag{4.34}$$

At last we obtain the system of equations for this case:

$$\begin{cases} \dfrac{1}{2}(m-1)(n-1)\,n\alpha_0+\gamma_1 c_i=\beta_1+\dfrac{2^{m+1-n}\delta_{i1}}{m!\,(n-1)^{m-1}} \\ (k-n)\,\alpha_{k-1}-k\alpha_k+\gamma_k c_i=\beta_k \qquad (k=1, ..., n-1) \\ \dfrac{1}{2}(m-1)(n-1)\,n\alpha_{n-1}+\gamma_{n-1}c_i=\beta_{n-1}+\dfrac{(-1)^{m(n-1)+1}\,2^{m+1-n}\delta_{in}}{m!\,(n-1)^{m-1}} \end{cases} \tag{4.35}$$

Expanding the coefficient determinant of this system in terms of the elements of the last column we get

$$\begin{aligned} A_n(-1,-1) &= \frac{1}{2}(m-1)(n-1)\,n\{[\gamma_1+(-1)^n\gamma_{n-1}](-1)^{n-1}(n-1)! \\ &\quad +\frac{1}{2}(m-1)(n-1)n\sum_{k=1}^{n-1}(-1)^{n-k}(k-1)!\,(n-k-1)!\,\gamma_k\} \\ &= (-1)^{n-1}2^{-n-1}(m-1)(n-1)n! \\ &\quad \times\left\{[1+(-1)^n](n-1)-\frac{1}{2}(m-1)(n-1)\sum_{k=1}^{n-1}(-1)^k\binom{n}{k}\right\} \\ &= (-1)^{n-1}2^{-n-2}(m^2-1)(n-1)^2\,n!\,[1+(-1)^n] \\ &= (-1)^{n-1}2^{-n-2}(m^2-1)(n-1)^2\,n!\,D_n(-1,-1). \end{aligned}$$

Obviously, if n is odd, then $D_n(-1,-1)=0$ and if n is even then $D_n(-1,-1)=2$.

Solving (4.35) by Cramer's rule for even n we obtain c_i. Meanwhile using $d_i=q_i(1)=2^{n-1}\alpha_0$ and (4.33) we give d_i.

This completes the proof.

COROLLARY. *Let* $\alpha, \beta > -1$. *The problem of* $(0, 1, ..., m-2, m)$*-interpolation is singular if one of the following conditions is satisfied*

(a) *Both* γ *and* δ *are integers, and* $n>\gamma+\delta$;

(b) $\alpha=\beta$ *and* n *is odd.*

The problem is regular if only one of γ *and* δ *is an integer and if* $n>\gamma+\delta$.

Proof. Case (a) has been shown in the proof of the theorem. Now let us show Case (b). Since

$$D_n(\alpha, \beta)=(-1)^n D_n(\beta, \alpha),$$

if n is odd and $\alpha=\beta$ then $D_n(\alpha, \alpha)=0$.

For the last conclusion we note that if, say, γ is an integer and δ is not, then

$$D_n(\alpha, \beta)=n! \sum_{k=0}^{\gamma} \frac{(-1)^k \binom{\gamma}{k}\binom{\delta}{n-k} \gamma_k}{\binom{n}{k}}.$$

But

$$\operatorname{sgn}\binom{\delta}{n-k}=(-1)^{n-k+[\delta]+1}.$$

So

$$\operatorname{sgn} D_{nn}(\alpha, \beta)=(-1)^{n+[\delta]+1}.$$

This completes the proof.

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